$K(\mathbb{Z})$

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This is a distillation of the results contained in [Weia] as they pertain specifically to the case of the integers. The classification is mod 8, and for even degrees mod 4 (that is the mod 8 classification has symmetry). Theorem 1 gives the odd mod 8 cases in terms of a mysterious integer w_i, for $K(\mathbb{Z})$ w_i is given by Lemma 27. The effort of unravelling this is to make sense of all indices. The end result is the following. Let $\mathcal{D}_n$ denote the denominator of $B_{(n+1)/4}/(n+1)$ where B_i is the i-th (topologists [Weia, Example 24]) Bernoulli number, then we have for the odd congruences mod 8

$$K_n(\mathbb{Z}) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}_2, & n \equiv 1 \\ \mathbb{Z}_{2\mathcal{D}_n}, & n \equiv 3 \\ \mathbb{Z}, & n \equiv 5 \\ \mathbb{Z}_{\mathcal{D}_n}, & n \equiv 7 \end{cases}$$

The even degree congruences are actually not known in general. We have the following facts in general

$$K_n(\mathbb{Z}) = \begin{cases} 0 \text{ iff Van Divers conjecture holds}, & n \equiv 0, 4 \\ \text{has order the numerator of } \frac{4B_{(n+2)/4}}{n+2}, & n \equiv 2 \\ \text{has order the numerator of } \frac{2B_{(n+2)/4}}{n+2}, & n \equiv 6 \end{cases}$$

Conjecturally these groups are simply cyclic of the orders listed in there. In particular this is known for the first like 20,000 K groups for $n \not\equiv 0, 4$. The first groups are presented in the following table from Weibel

Table 5.1. The groups $K_n(\mathbb{Z})$, $n < 20\,000$. The notation '(0?)' refers to a finite group, conjecturally zero, whose order is a product of irregular primes $> 10^7$

$K_0(\mathbb{Z}) = \mathbb{Z}$	$K_8(\mathbb{Z}) = (0?)$	$K_{16}(\mathbb{Z}) = (0?)$
$K_1(\mathbb{Z}) = \mathbb{Z}/2$	$K_9(\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}/2$	$K_{17}(\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}/2$
$K_2(\mathbb{Z}) = \mathbb{Z}/2$	$K_{10}(\mathbb{Z}) = \mathbb{Z}/2$	$K_{18}(\mathbb{Z}) = \mathbb{Z}/2$
$K_3(\mathbb{Z}) = \mathbb{Z}/48$	$K_{11}(\mathbb{Z}) = \mathbb{Z}/1008$	$K_{19}(\mathbb{Z}) = \mathbb{Z}/528$
$K_4(\mathbb{Z}) = 0$	$K_{12}(\mathbb{Z}) = (0?)$	$K_{20}(\mathbb{Z}) = (0?)$
$K_5(\mathbb{Z}) = \mathbb{Z}$	$K_{13}(\mathbb{Z}) = \mathbb{Z}$	$K_{21}(\mathbb{Z}) = \mathbb{Z}$
$K_6(\mathbb{Z}) = 0$	$K_{14}(\mathbb{Z}) = 0$	$K_{22}(\mathbb{Z}) = \mathbb{Z}/691$
$K_7(\mathbb{Z}) = \mathbb{Z}/240$	$K_{15}(\mathbb{Z}) = \mathbb{Z}/480$	$K_{23}(\mathbb{Z}) = \mathbb{Z}/65\,520$

Remark. The K groups of the integers are computable in general Reference?. They have in practice been computed up to about $20,000$. Thus we know a good deal about the first $20,000$ even degrees. Again for $n \not\equiv 0, 4$. In this regard however there are recent results for instance showing that $K_8(\mathbb{Z}) = 0$.

Remark. Just to be ultra clear $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$.

Remark. Here are some remarks on the even groups proofs. First the universal coefficient theorem for K groups [Weib, IV.2.5] makes it possible to go from K_n with coefficients to the local peices of K_n. Then there is a spectral sequence from some cohomology called "Motivic cohomology" which describes the K groups with torsion coefficients up to an extension problem, hence we can deduce the order from this information. Finally the motivic cohomology groups is isomorphic in a range to certain Etale cohomology groups that can be computed by Galois theory.

Remark. FLT follows in a non-trivial way from Van-Divers conjecture. Thus computing all the K groups of the integers would prove FLT.

References

[Weia] Charles Weibel. Algebraic K-Theory I.5 of Rings of Integers in Local and Global Fields.

[Weib] Charles A Weibel. The K-book an introduction to Algebraic K-theory.